MATH 2500 DERIVATIVE PROPERTIES

Name:____

DEFINITION: Given a function f, the derivative of f, denoted f', is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists as a real number. If $f'(a)$ exists, then we say f is '*differentiable*' at $x = a$.

NOTATIONS: If $y = f(x)$: $f'(x)$, y', $\dot{y}$, $\frac{dy}{dx}$, $\frac{d}{dx}[f(x)]$, and $D_x[f(x)]$ all represent the derivative.

GRAPHICAL INTERPRETATION: Geometrically, $f'(a)$ represents the slope of the *tangent line* to the graph $y = f(x)$ at the point $(a, f(a))$: $y = f(a) + f'(a)(x - a)$. If you imagine graphing a curve on your calculator and repeatedly zooming in on a point, differentiable functions will appear indistinguishable from a line - this is the tangent line. This 'locally linear' property can be formalized as follows:

THEOREM: A function f is differentiable at $x = a$ means there is a function $\epsilon(x)$ so that:

$$f(x) = f(a) + f'(a)(x - a) + \epsilon(x)(x - a),$$

where $\lim_{x \to a} \epsilon(x) = 0$. That is, for x 'near' a, $f(x) \approx f(a) + f'(a)(x - a)$.

RATE OF CHANGE: If $y = f(x)$, we can compute the rate of change of y with respect to x:

$\frac{\Delta y}{\Delta x}$ is the *average* rate of change; $\frac{dy}{dx}$ is the *instantaneous* rate of change.

THEOREM: If f is differentiable at $x = a$, then f is continuous at $x = a$.

NOTE: If f is continuous at $x = a$, then f **isn't necessarily differentiable** at $x = a$. Consider the following functions at $x = 0$: $f(x) = |x|$ (a 'corner'), $f(x) = \sqrt[3]{x}$ (a 'vertical tangent'), and $f(x) = x^{2/3}$ (a 'cusp.')

PROPERTIES OF THE DERIVATIVE:

Suppose f and g are differentiable and k is a real number constant.

- **The Constant Rule**: $D_x[k] = 0$
- **The Power Rule**: $D_x[x] = 1$. More generally, $D_x[x^k] = kx^{k-1}$ for $k \neq 0$.
- **The Constant Multiple Rule**: $D_x[kf(x)] = kD_x[f(x)] = kf'(x)$.
- **The Sum/Difference Rule**: $D_x[f(x) \pm g(x)] = D_x[f(x)] \pm D_x[g(x)] = f'(x) \pm g'(x)$.
- **The Product Rule**: $D_x[f(x)g(x)] = D_x[f(x)]g(x) + f(x)D_x[g(x)] = f'(x)g(x) + f(x)g'(x)$.
- **The Quotient Rule**: $D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)D_x[f(x)] - f(x)D_x[g(x)]}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

DERIVATIVES OF THE CIRCULAR FUNCTIONS:

- $D_x[\sin(x)] = \cos(x)$
- $D_x[\sec(x)] = \sec(x)\tan(x)$
- $D_x[\tan(x)] = \sec^2(x)$
- $D_x[\cos(x)] = -\sin(x)$
- $D_x[\csc(x)] = -\csc(x)\cot(x)$
- $D_x[\cot(x)] = -\csc^2(x)$

THE CHAIN RULE: If f is differentiable at $g(x)$ and g is differentiable at x, then: $D_x\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$.
Said differently: if f is a differentiable function of u and u is a differentiable function of x, then:

$$D_x\left[f(u)\right] = f'(u) \cdot u' \text{ or, equivalently: } \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.$$

GENERALIZED DERIVATIVE FORMULAS:

- **POWER RULE:** $D_x[u^k] = ku^{k-1} \cdot u'$
- $D_x[\sin(u)] = \cos(u) \cdot u'$
- $D_x[\cos(u)] = -\sin(u) \cdot u'$
- $D_x[\sec(u)] = \sec(u)\tan(u) \cdot u'$
- $D_x[\csc(u)] = -\csc(u)\cot(u) \cdot u'$
- $D_x[\tan(u)] = \sec^2(u) \cdot u'$
- $D_x[\cot(u)] = -\csc^2(u) \cdot u'$